Project 3 – MANE 6710 Numerical Design Optimization

*Unique ID: 5284*

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# Introduction

The main objective of this project is to achieve optimal parameters to design Tilt-A-Whirl. Tilt-A-Whirl is a theme park joy ride which has its physics based on deterministic chaos. People going on this ride have a lot of fun because they cannot anticipate their motion throughout the ride. This project focusses on achieving design parameters of Tilt-A-Whirl mechanism that can maximize this chaotic behavior, enabling people taking the ride to have utmost fun.

# Deterministic Chaos

It is important to provide an explanation about deterministic chaos before further work is discussed in detail. A dynamical system expressed as Ordinary Differential Equations (ODEs), is the rate of change of certain “state” variables with respect to time governed by underlying physics and deterministic laws. This dynamical system becomes chaotic if there is no way to predict what or how the state variables will behave over a long period of time. Tilt-A-Whirl extracts all its fun elements from this very concept of unpredictability.

Tilt-A-Whirl design [1] is parametrized by several variables, and it is designed as shown in Figure 1.

Diagram

Description automatically generated

Figure 1. Top View of Tilt-A-Whirl platform

Work done in [1], derives the equations of motion for the system described in Figure 2. Various Orientations of Tilt-A-Whirl design and the variables used are described in Table 1.

|  |  |  |
| --- | --- | --- |
| **Variable** | **Description** | **Units** |
|  | Distance from axis to center of platform | m |
|  | Distance from center of platform to car | m |
|  | Location of platform and angular velocity | rad, 1/s |
|  | Position of car and its angular velocity | rad, 1/s |
|  |  | No dim |
|  | Non dimensional term | No dim |
|  | Global coordinate system | NA |
|  | Local coordinate system of each car | NA |
|  | Angles of w.r.t | rad |
|  | , t - seconds | No dim |

Table 1. Variable Description

Diagram, engineering drawing

Description automatically generated

Figure 2. Various Orientations of Tilt-A-Whirl design

The equation of motion is:

|  |  |
| --- | --- |
|  | Eqn 1 |

|  |  |
| --- | --- |
|  | Eqn 2 |
|  | Eqn 3 |
|  | Eqn 4 |
|  | Eqn 5 |
|  | Eqn 6 |

Here, – mass and is the damping constant. ensures the system is over-damped making the user experience fun as well as safe.

## Simulating Deterministic Chaos

The next step is to simulate Eqn 1, using the state-space method of re-writing this equation.

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  | Eqn 7 |

This form of the equation as derived in Eqn 7 can be simulated using MATLAB’s ODE simulation function *ODE45(@chaosDynamics, Time Range, Starting point).* This command takes function handle @chaosDynamics (Eqn 7), the time range (non-dimensional time) and the starting point of the states . For starting points and and a non-dimensional time the states vary as shown in Figure 3.

Figure 3. a) Starting point , b) Starting point

|  |  |
| --- | --- |
| Diagram  Description automatically generateda) | Diagram  Description automatically generatedb) |

Table 2. shows the list of parameters which are fixed as design constraints and those that cannot be varied. From Figure 3., it can be clearly seen that for different starting points and these set of parameters, the trajectories of , vary significantly over time. This is because of the chaotic nature of the dynamical system. The most important point to be noted is the noisy nature of how varies with respect to time.

Table 2. a) Constant parameters. b) Design Variables and their range

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Parameters | Values | Units |  | Design Vars | low | high |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |

is the angular velocity of the car and passengers will have fun if it gets noisier.

# Objective function

## Choice of Design Variables

Table 2.a) lists all the parameters which are constant from a design perspective. So, parameters are the remaining variables which can be used to modify the behavior of with respect to . These are the choice of design variables in this project. Table 2.b) lists the design variables and their permissible range of values.

## Formulation of Objective function

The objective is to have maximum possible chaotic behavior in which means, it is important to achieve high variance or standard deviation of the trajectory of over large intervals of time.

|  |  |
| --- | --- |
|  |  |
|  | Eqn 8 |
|  |  |

Where, is the total simulation time and . A one-dimensional slice of this objective function with fixed and varying over its permissible range of values is shown in Figure 4. This is proof of how noisy the actual objective function is. A gradient based optimizer will find it difficult to get to an optimal point due to the extremely spiky objective function and it will get lost in one of its multiple local minimizers. Hence, it is important to get a smooth representation of the objective function in order to find an optimal point.

Chart, histogram

Description automatically generated

Figure 4. 1-D slice of objective function along omega

# Surrogate Model

The next step in this project is to generate a surrogate model that presents a smooth representation of this noisy objective function behavior. Gaussian Process Library (GPML) is used to generate the surrogate model. Sample points are generated by the Lattice Hypercube Sampling technique and the true objective function values are computed for each of those sample points. The true objective values at these sample locations are labelled as the training data and it drives the process of generating a Gaussian process surrogate. 1000 points have been sampled within the range of permissible values of the design variables. is the function from GPML library which can be used to perform prediction using the trained parameters. The trained hyperparameters are shown in Table 3.

Table 3. Trained Hyperparameters

|  |  |
| --- | --- |
| Covariance |  |
| Likelihood |  |

Chart, line chart, histogram

Description automatically generated

Figure 5. Surrogate model vs True Objective comparison

Figure 5. shows how the surrogate prediction of the 1-D slice of the objective function compared to the true objective function. It can be seen that the prediction is a smooth representation of the chaotic, noisy true objective function. The surrogate can be used as an alternative to the true objective function in the optimization process if it is an accurate enough approximation.

## Accuracy of Surrogate Model

It is important to have an accurate surrogate that represents the true objective function or else the gradient based optimizer will lead to an incorrect optimal point. The accuracy is checked by computing the mean squared error between the prediction and the true objective of the 1-D slice in consideration.

|  |  |
| --- | --- |
|  | Eqn 9 |

To do this analysis, a lot of variables can be used to compute the accuracy of the surrogate, but the choice of variables is number of sample data and the integration time period of the differential equation.

Chart, line chart

Description automatically generated

Figure 6. Mean Square Error for 1-D slice of objective v Surrogate

The least error between the surrogate and the true objective function was observed when the number of bins used were and a total non-dimensional integration time of the chaotic differential equation is . But this is a very small integration period to truly enjoy the chaotic nature of the ride. For a maximum non-dimensional integration time of , the least MSE was observed when the number of bins used in LHS was . The fit of surrogate model with the true objective for, , is shown in Figure 7.

Chart, line chart

Description automatically generated

Figure 7. tau = 500, bins = 1000 surrogate performance

# Optimization setup

With the surrogate model up and setup, the optimization statement can be written as:

|  |  |
| --- | --- |
|  | Eqn 10 |

where, and the smooth surrogate approximation is used as the objective function. The range of the design variables are the lower and upper bound constraints. MATLAB’s inbuilt ***fmincon(obj,x0,A,b,Aeq,beq,lb,ub,nonlincon,opt)*** is used for performing the optimization and find the optimal maximizer.

# Results

For a starting point of and a non-dimensional integration period of , the objective at the starting point is . A surrogate trained with for the same , the optimal point achieved is

.

The **optimal standard deviation achieved using the surrogate is** .

This result translates into this following physical intuition of the problem statement described in Table 4

Table 4. Physical Intuition of results

|  |  |  |  |
| --- | --- | --- | --- |
| Variable | Initial point | Optimal point | Physical Intuition |
|  |  |  | Decrease in , is a rise in , which implies a reduction of torque required to spin the car around on the platform |
|  |  |  | Increase in directly affects the angles (angle of ). It makes the ride a lot bumpier compared to the flatter initial design. |
|  |  |  | A rise in is a clear indication of the fact that, a faster ride has direct correlation to increased fun. |

**Optimal standard deviation achieved using ODE45 is** .This is a result which aligns with the physical intuition of the system dynamics and shows an improvement of approximately from the original objective. This also shows that the surrogate is a good approximation for the real deterministic chaotic simulation.

The First-Order optimality plot shown in Figure 8. is not a traditional looking plot because, the surrogate objective is trying fit the true objective which is extremely noisy. This makes the surrogate also a little noisy and that is reflected in Figure 8.

Figure 8. First Order Optimality plot Chart, line chart

Description automatically generated

Chart, line chart

Description automatically generated

Figure 9. Convergence Analysis plot

## Convergence Analysis

Increasing the non-dimensional integration time is equivalent to increasing the integration time period of the ODE as they are directly proportional to each other. With an increase in the integration period, a rise in the objective value is noticed and it converges around the band of values of standard deviation. Since the true objective is a noisy function, the optimal function values converging to a band will not be a surprise given its chaotic nature. This can be clearly seen in Figure 9.

Figure 10. Angular velocity comparison between starting and optimal pointA picture containing graphical user interface

Description automatically generated

The red curve is the trajectory of with initial parameters. The blue curve is the trajectory of with optimal design parameters.

# Works Cited

|  |  |
| --- | --- |
| [1] | B. M. R.L.Kautz, "Chaos at the amusement park: Dynamics of the Tilt-A-Whirl," *American Association of Physics Teachers,* vol. 62, no. 1, pp. 59-65, 29 June 1993. |